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Approximation of functions of two variables by certain linear positive operators

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Abstract. We introduce certain linear positive operators and study some approximation properties of these operators in the space of functions, continuous on a compact set, of two variables. We also find the order of this approximation by using modulus of continuity. Moreover we define an rth order generalization of these operators and observe its approximation properties. Furthermore, we study the convergence of the linear positive operators in a weighted space of functions of two variables and find the rate of this convergence using weighted modulus of continuity.

Keywords. Linear positive operator; modulus of continuity; order of approximation; polynomial weighted spaces.

1. Introduction

Let $f \in C([0,1])$. The well-known Bernstein polynomial of degree n, denoted by $B_n(f;x)$ is

$$B_n(f;x) := \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad n \in \mathbb{N} = \{1,2,\dots\},$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \tag{1.1}$$

and $x \in [0,1]$ [4].

Let $x \in [0,\infty)$ and $f \in C([0,\infty))$. Szazs–Mirakyan operators, denoted by $S_n(f;x)$ are

$$S_n(f;x) := \sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{k}{n}\right), \quad n \in \mathbb{N},$$

where

$$q_{n,k}(x) = \mathrm{e}^{-nx} \frac{(nx)^k}{k!}. \tag{1.2}$$

In [3] approximation properties of $S_n(f;x)$ in weighted spaces were studied. Some works by Szazs–Mirakyan or modified Szazs–Mirakyan operators may be found in [12,7,18] and references therein.

Stancu [16] introduced the following generalization of the Bernstein polynomials. Let $f \in C([0,1])$. Stancu operators, denoted by $(P_n^{(\alpha,\beta)} f)$ are

$$(P_n^{(\alpha,\beta)} f) := \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right), \quad n \in \mathbb{N},$$

where $p_{n,k}(x)$ are the polynomials given by (1.1), α, β are positive real numbers satisfying $0 \le \alpha \le \beta$.

Taking the operators, given above, into account we now introduce certain linear positive operators of functions of two variables as follows:

Let $f \in C(\mathscr{R}), \mathscr{R} := [0,1] \times [0,\infty)$ and let the linear positive operators $L_{m,n}^{\alpha_i,\beta_j}$, $j = 1,2$, be defined as follows:

$$L_{m,n}^{\alpha_i,\beta_j} := \sum_{k=0}^{\infty} \sum_{\nu=0}^{m} p_{m,\nu}(x) q_{n,k}(y) f\left(\frac{\nu+\alpha_1}{m+\beta_1}, \frac{k+\alpha_2}{n+\beta_2}\right) \tag{1.3}$$

for $(x,y) \in \mathscr{R}, m,n \in \mathbb{N}, 0 \le \alpha_j \le \beta_j$, $j = 1,2$, where $p_{m,\nu}(x)$ and $q_{n,k}(y)$ are given in (1.1) and (1.2), respectively. In the sequel, whenever we mention the operators $L_{m,n}^{\alpha_i,\beta_j}$, $j = 1,2$, it will be mentioned that these are the operators given in (1.3). We use the notation $\mathscr{R}_A$ to denote the following closed and bounded region in $\mathbb{R}^2$,

$$\mathscr{R}_A := [0,1] \times [0,A], \quad A > 0. \tag{1.4}$$

In this paper we first study some approximation properties of the sequence of linear positive operators given by (1.3) in the space of functions, continuous on $\mathscr{R}_A$, and find the order of this approximation using modulus of continuity. Moreover we define an rth order generalization of $L_{m,n}^{\alpha_i,\beta_j}$, $j = 1,2$, on $\mathscr{R}_A$ extending the results of Kirov [14] and Kirov–Popova [15] to the linear positive operators $L_{m,n}^{\alpha_i,\beta_j}$, $j = 1,2$, of functions of two variables and study its approximation properties. The rth order generalization of some kind of linear positive operators may also be found in [1,9].

We finally investigate the convergence of the sequence of linear positive operators $L_{m,n}^{\alpha_i,\beta_j}$, $j = 1,2$, defined on a weighted space of functions of two variables and find the rate of this convergence by means of weighted modulus of continuity.

If we take $p_{n,k}(y), k = 0,1,\ldots,n$ in place of $q_{n,k}(y)$ in (1.3), then the operators $L_{m,n}^{\alpha_i,\beta_j}$, $j = 1,2$, reduce to the generalized Bernstein polynomials of two variables which were studied in [5].

Approximation of functions of one or two variables by some positive linear operators in weighted spaces may be found in [8,9,13,17,18].

2. Preliminaries

In this section we give some basic definitions which we shall use. We denote by ρ the function, continuous and satisfying $\rho(x,y) \geqslant 1$ for $(x,y) \in \mathscr{R}$ and $\lim_{|r|\to\infty} \rho(x,y) = \infty, r = (x,y) \cdot \rho$ is called a weight function. Let B_ρ denote the set of functions of two variables defined on $\mathscr{R}$ satisfying $|f(x,y)| \leq M_f \rho(x,y)$, where $M_f > 0$ is a constant depending on f, and C_ρ denote the set of functions belonging to B_ρ, and continuous on $\mathscr{R}$. Clearly $C_\rho \subset B_\rho \cdot B_\rho$ and C_ρ are called weighted spaces with norm $\|f\|_\rho = \sup_{(x,y)\in\mathscr{R}} \frac{|f(x,y)|}{\rho(x,y)}$, [10,11].

The Lipschitz class $\mathrm{Lip}_M(\gamma)$ of the functions of f of two variables is given by

$$|f(x_1,y_1) - f(x_2,y_2)| \leq M[(x_1-x_2)^2 + (y_1-y_2)^2]^{\frac{\gamma}{2}}, \tag{2.1}$$

$(x_1,y_1),(x_2,y_2) \in \mathscr{R}$, where $M > 0, 0 < \gamma \leq 1$ and $f \in C(\mathscr{R})$. The full modulus of continuity of $f \in C(\mathscr{R}_A)$, denoted by $w(f;\delta)$, is defined as follows:

$$w(f;\delta) = \max_{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \leq \delta} |f(x_1,y_1) - f(x_2,y_2)|. \tag{2.2}$$

Partial modulus of continuity with respect to x and y are given by

$$w^{(1)}(f;\delta) = \max_{0\leq y\leq A} \max_{|x_1-x_2|\leq\delta} |f(x_1,y) - f(x_2,y)| \tag{2.3}$$

and

$$w^{(2)}(f;\delta) = \max_{0\leq x\leq 1} \max_{|y_1-y_2|\leq\delta} |f(x,y_1) - f(x,y_2)|, \tag{2.4}$$

respectively. We shall also need the following properties of the full and partial modulus of continuity

$$w(f;\lambda\delta) \leq (1+[\lambda])w(f;\delta) \tag{2.5}$$

for any λ. Here $[\lambda]$ is the greatest integer that does not exceed λ. Moreover, it is known that when f is uniformly continuous, then $\lim_{\delta\to 0} w(f;\delta) = 0$ and

$$|f(t,\tau) - f(x,y)| \leq w(f;\sqrt{(t-x)^2 + (\tau-y)^2}), \tag{2.6}$$

$(t,\tau),(x,y) \in \mathscr{R}_A$. The analogous properties are satisfied by the partial modulus of continuity.

3. Lemmas and theorems on $\mathscr{R}_A$

In this section we give some classical approximation properties of the operators $L_{m,n}^{\alpha_i,\beta_j}$, $j = 1,2$, on the compact set $\mathscr{R}_A$.

Lemma 3.1. *Let α_j, β_j, $j = 1,2$, be the fixed positive numbers such that $0 \leq \alpha_j \leq \beta_j$. Then we have*

$$L_{m,n}^{\alpha_i,\beta_j}(1;x,y) = 1,$$

$$L_{m,n}^{\alpha_i,\beta_j}(t;x,y)=\frac{mx+\alpha_1}{m+\beta_1},$$

$$L_{m,n}^{\alpha_i,\beta_j}(\tau;x,y)=\frac{nx+\alpha_2}{n+\beta_2},$$

$$L_{m,n}^{\alpha_i,\beta_j}(t^2+\tau^2;x,y)=\frac{(m^2-m)x^2+(2\alpha_1+1)mx+\alpha_1^2}{(m+\beta_1)^2}+\frac{n^2y^2+(2\alpha_2+1)ny+\alpha_2^2}{(n+\beta_2)^2}$$

for all $m,n\in N$.

Taking (3.1) into account we now give the following Baskakov type theorem (see [2] to get the approximation to $f(x,y)\in C(\mathscr{R}_A)$, satisfying $|f(x,y)|\le M_f(1+x^2+y^2)$, by $L_{m,n}^{\alpha_i,\beta_j}$, $j=1,2$.

Theorem 3.2. *Let $f(x,y)\in C(\mathscr{R}_A)$ and $|f(x,y)|\le M_f(1+x^2+y^2)$ for $(x,y)\in R$. Here M_f is a constant depending on f. Then $\|L_{m,n}^{\alpha_i,\beta_j}(f;x,y)-f(x,y)\|_{C(\mathscr{R}_A)}\to 0$, as $m,n\to\infty$ if and only if*

$$\|L_{m,n}^{\alpha_i,\beta_j}(1;x,y)-1\|_{C(\mathscr{R}_A)}\to 0,$$

$$\|L_{m,n}^{\alpha_i,\beta_j}(t;x,y)-x\|_{C(\mathscr{R}_A)}\to 0,$$

$$\|L_{m,n}^{\alpha_i,\beta_j}(\tau;x,y)-y\|_{C(\mathscr{R}_A)}\to 0,$$

$$\|L_{m,n}^{\alpha_i,\beta_j}(t^2+\tau^2;x,y)-(x^2+y^2)\|_{C(\mathscr{R}_A)}\to 0, \tag{3.1}$$

as $m,n\to\infty$ for $(x,y)\in R_A$.

Proof. Since the necessity is clear, then we need only to prove the sufficiency. Let $(t,\tau),(x,y)\in\mathscr{R}_A$. By the uniform continuity of f on $\mathscr{R}_A$ we get that for each $\varepsilon>0$ there exists a number $\delta>0$ such that $|f(t,\tau)-f(x,y)|<\varepsilon$, whenever $\sqrt{(t-x)^2+(\tau-y)^2}<\delta$. Now let $(x,y)\in\mathscr{R}_A$ and $(t,\tau)\in\mathscr{R}$ and let (x_1,y_1) be an arbitrary boundary point of $\mathscr{R}_A$ such that $0\le x_1\le 1, 0\le y_1\le A$. Since f is continuous on the boundary points also, then for each $\varepsilon>0$ there exists a $\delta>0$ such that

$$|f(t,\tau)-f(x,y)|\le|f(t,\tau)-f(x_1,y_1)|+|f(x_1,y_1)-f(x,y)|<\varepsilon$$

whenever $\sqrt{(t-x)^2+(\tau-y)^2}<\delta$. Finally let $(x,y)\in\mathscr{R}_A$ and $(t,\tau)\in\mathscr{R}$ and let $\sqrt{(t-x)^2+(\tau-y)^2}>\delta$. Then easy calculations show that

$$\begin{aligned}|f(t,\tau)-f(x,y)|&\le M_f((t-x)^2+(\tau-y)^2)\left(\frac{2}{\delta^2}+2+\frac{3}{\delta^2}(x^2+y^2)\right)\\&\le C\left(\frac{(t-x)^2+(\tau-y)^2}{\delta^2}\right).\end{aligned}$$

Here $C > 0$ is a constant. Therefore we get

$$|f(t,\tau) - f(x,y)| \le \varepsilon + C\left(\frac{(t-x)^2 + (\tau-y)^2}{\delta^2}\right), \tag{3.2}$$

for $(t,\tau) \in \mathscr{R}, (x,y) \in \mathscr{R}_A$. Applying $L_{m,n}^{\alpha_i,\beta_j}$ to (3.2) we get

$$|L_{m,n}^{\alpha_i,\beta_j}(f(t,\tau);x,y) - f(x,y)| \le L_{m,n}^{\alpha_i,\beta_j}(|f(t,\tau) - f(x,y)|;x,y)$$
$$+ \|f\||L_{m,n}^{\alpha_i,\beta_j}(1;x,y) - 1|.$$

Using (3.2) in the last inequality and taking (3.1) into account, sufficiency is obtained easily. □

We note that if we take $f(x,y)$ to be bounded on $\mathbb{R}^2$ in the previous theorem, then we easily obtain that $\|L_{m,n}^{\alpha_i,\beta_j}(f;x,y) - f(x,y)\|_{C(\mathscr{R}_A)} \to 0$, as $m,n \to \infty$ satisfied from Lemma 3.1 by analogous Korovkin's theorem proved by Volkov [19].

The following theorem gives the rate of convergence of the sequence of linear positive operators $\{L_{m,n}^{\alpha_i,\beta_j}\}$ to f, by means of partial and full modulus of continuity.

Theorem 3.3. *Let $f \in C(\mathscr{R}_A)$. Then the following inequalities*

(a) $$\|L_{m,n}^{\alpha_i,\beta_j}(f;x,y) - f(x,y)\|_{C(\mathscr{R}_A)} \le \frac{3}{2}\{w^{(1)}(f;\delta_m) + w^{(2)}(f;\delta_n)\}, \tag{3.3}$$

(b) $$\|L_{m,n}^{\alpha_i,\beta_j}(f;x,y) - f(x,y)\|_{C(\mathscr{R}_A)} \le \frac{3}{2}w(f;\delta_{m,n}) \tag{3.4}$$

hold, where R_A is the closed and bounded region given by (1.4). *$w^{(1)}, w^{(2)}$ and w are given by* (2.3), (2.4) *and* (2.2) *respectively, and $\delta_m, \delta_n, \delta_{m,n}$ are*

$$\delta_m = \frac{\sqrt{4\beta_1^2 + m}}{m + \beta_1}, \quad \delta_n = \frac{\sqrt{\beta_2^2 A^2 + nA}}{n + \beta_2}, \quad \delta_{m,n} = \sqrt{\delta_m^2 + 4\delta_n^2}, \tag{3.5}$$

respectively.

Proof. From (1.3) we have

$$|L_{m,n}^{\alpha_i,\beta_j}(f(t,\tau);x,y) - f(x,y)| \le \mathrm{e}^{-ny}\sum_{k=0}^{\infty}\sum_{\nu=0}^{m}\binom{m}{\nu}x^{\nu}(1-x)^{m-\nu}\frac{(ny)^k}{k!}$$
$$\times\left|f\left(\frac{\nu+\alpha_1}{m+\beta_1}, \frac{k+\alpha_2}{n+\beta_2}\right) - f(x,y)\right|. \tag{3.6}$$

Let us first add and drop the function $f\left(\frac{\nu+\alpha_1}{m+\beta_1}, y\right)$ inside the absolute value sign on the right-hand side of (3.6). Using the analogous property of (2.6) for the partial modulus of continuity and finally applying the Cauchy–Schwartz inequality to the resulting term, then we arrive at (3.3) on $\mathscr{R}_A$, which proves (a). Using (2.6) directly in (3.6) and applying Cauchy–Schwartz inequality to the resulting term we then reach to (3.4) on $\mathscr{R}_A$, which gives (b). □

COROLLARY 3.4.

Let $f \in \mathrm{Lip}_M(\gamma)$. *Then the inequality*

$$|L_{m,n}^{\alpha_i,\beta_j}(f;x,y) - f(x,y)| \leq M_1' \delta_{m,n}^{\gamma}$$

holds, where $M_1' = \frac{3}{2}M_1, M_1 > 0$ *and* $\delta_{m,n}$ *is given in* (3.5).

COROLLARY 3.5.

If f *satisfies the following Lipschitz conditions*

$$|f(x_1,y) - f(x_2,y)| \leq M_2|x_1 - x_2|^{\alpha}$$

and

$$|f(x,y_1) - f(x,y_2)| \leq M_3|y_1 - y_2|^{\beta},$$

$0 < \alpha,\beta \leq 1, M_j > 0, j = 2,3,$ *then the inequality*

$$|L_{m,n}^{\alpha_i,\beta_j}(f;x,y) - f(x,y)| \leq M_2' \delta_m^{\alpha} + M_3'(2\delta_n)^{\beta}$$

holds, where $M_2' = \frac{3}{2}M_2$ *and* $M_3' = \frac{3}{2}M_3$, *and* δ_m, δ_n *are given in* (3.5).

4. A generalization of order r of $L_{m,n}^{\alpha_i,\beta_j}$

Let $C^r(\mathscr{R}_A)$, $r \in \mathbb{N} \cup \{0\}$, denote the set of all functions f having all continuous partial derivatives up to order r at $(x,y) \in \mathscr{R}_A$.

By $(L_{m,n}^{\alpha_i,\beta_j})^{[r]}$, $j = 1,2$, we denote the following generalization of $L_{m,n}^{\alpha_i,\beta_j}$:

$$(L_{m,n}^{\alpha_i,\beta_j})^{[r]}(f;x,y) := \mathrm{e}^{-ny} \sum_{k=0}^{\infty} \sum_{\nu=0}^{m} \binom{m}{\nu} x^{\nu}(1-x)^{m-\nu} \frac{(ny)^k}{k!} \times P_{r,\left(\frac{\nu+\alpha_1}{m+\beta_1},\frac{k+\alpha_2}{n+\beta_2}\right)}\left(x - \frac{\nu+\alpha_1}{m+\beta_1}, y - \frac{k+\alpha_2}{n+\beta_2}\right), \tag{4.1}$$

$0 \leq \alpha_j \leq \beta_j$, $j = 1,2$, where

$$P_{r,\left(\frac{\nu+\alpha_1}{m+\beta_1},\frac{k+\alpha_2}{n+\beta_2}\right)}\left(x - \frac{\nu+\alpha_1}{m+\beta_1}, y - \frac{k+\alpha_2}{n+\beta_2}\right) = \sum_{h=0}^{r} \sum_{i+j=h} \frac{1}{h!} \binom{h}{j} f_{x^i y^j}\left(\frac{\nu+\alpha_1}{m+\beta_1}, \frac{k+\alpha_2}{n+\beta_2}\right) \times \left[x - \frac{\nu+\alpha_1}{m+\beta_1}\right]^i \left[y - \frac{k+\alpha_2}{n+\beta_2}\right]^j, \tag{4.2}$$

$f_{x^i y^j}$ denotes the partial derivatives of f, i.e.: $f_{x^i y^j} := \frac{\partial^r}{\partial x^i \partial y^j} f(x,y)$.

$(L_{m,n}^{\alpha_i,\beta_j})^{[r]}$ are called the rth order of $L_{m,n}^{\alpha_i,\beta_j}$ (see [14] for one variable). Obviously $(L_{m,n}^{\alpha_i,\beta_j})^{[r]}$ reduce to $L_{m,n}^{\alpha_i,\beta_j}$, when $r = 0$.

Now let us write

$$\left(x - \frac{\nu+\alpha_1}{m+\beta_1}, y - \frac{k+\alpha_2}{n+\beta_2}\right) = u(\alpha,\beta), \tag{4.3}$$

where (α,β) is a unit vector, $u > 0$ and let

$$\begin{aligned} F(u) &= f\left(\frac{\nu+\alpha_1}{m+\beta_1} + u\alpha, \frac{k+\alpha_2}{n+\beta_2} + u\beta\right) \\ &= f\left[\frac{\nu+\alpha_1}{m+\beta_1} + \left(x - \frac{\nu+\alpha_1}{m+\beta_1}\right), \frac{k+\alpha_2}{n+\beta_2} + \left(y - \frac{k+\alpha_2}{n+\beta_2}\right)\right]. \end{aligned} \tag{4.4}$$

It is clear that Taylor's formula for $F(u)$ at $u = 0$ turns into Taylor's formula for $f(x,y)$ at $\left(\frac{\nu+\alpha_1}{m+\beta_1}, \frac{k+\alpha_2}{n+\beta_2}\right)$. Morever rth derivative takes the form

$$F^{(r)}(u) = \sum_{i+j=r} \binom{r}{j} f_{x^i y^j} \left(\frac{\nu+\alpha_1}{m+\beta_1} + u\alpha, \frac{k+\alpha_2}{n+\beta_2} + u\beta\right) \alpha^i \beta^j, \tag{4.5}$$

$r \in \mathbb{N}$ (see Chapter 3 of [6]).

By means of the modification stated above ((4.3)–(4.5)), we get the following result.

Theorem 4.1. *Let $f \in C^r(\mathscr{R}_A)$ and $F^{(r)}(u) \in \mathrm{Lip}_M(\gamma)$. Then the inequality*

$$\begin{aligned} &\|(L_{m,n}^{\alpha_i,\beta_j})^{[r]}(f;x,y) - f(x,y)\|_{C(\mathscr{R}_A)} \\ &\quad \le \frac{\gamma M}{\gamma + r} \frac{B(\gamma,r)}{(r-1)!} \|L_{m,n}^{\alpha_i,\beta_j}(|(x,y) - (t,\tau)|^{r+\gamma}; x,y)\|_{C(\mathscr{R}_A)} \end{aligned} \tag{4.6}$$

holds, where $F^{(r)}(u)$ are given by (4.5), $B(\gamma,r)$ is the well-known beta function, $r,m,n \in N$, $0 < \gamma \le 1$ and $M > 0$.

Proof. From (4.1) and (4.2) we have

$$\begin{aligned} &f(x,y) - (L_{m,n}^{\alpha_i,\beta_j})^{[r]}(f;x,y) \\ &\quad = \sum_{\nu=0}^{m} \binom{m}{\nu} x^\nu (1-x)^{m-\nu} \mathrm{e}^{-ny\frac{(ny)^k}{k!}} \\ &\qquad \times \sum_{k=0}^{\infty} \left\{ f(x,y) - \sum_{h=0}^{r} \frac{1}{h!} \sum_{i+j=h} \binom{h}{j} f_{x^i y^j} \right. \\ &\qquad \left. \times \left(\frac{\nu+\alpha_1}{m+\beta_1}, \frac{k+\alpha_2}{n+\beta_2}\right) \left[x - \frac{\nu+\alpha_1}{m+\beta_1}\right]^i \left[y - \frac{k+\alpha_2}{n+\beta_2}\right]^j \right\}. \end{aligned} \tag{4.7}$$

We now consider Taylor's formula with the remainder for the functions of two variables. Using the integral form of the remainder term that appeared in (4.7), we arrive at

$$\begin{aligned}
&f(x,y) - P_{r,\left(\frac{\nu+\alpha_1}{m+\beta_1},\frac{k+\alpha_2}{n+\beta_2}\right)}\left(x-\frac{\nu+\alpha_1}{m+\beta_1}, y-\frac{k+\alpha_2}{n+\beta_2}\right)\\
&= \frac{1}{(r-1)!}\int_0^1 \sum_{i+j=h}\binom{h}{j}\left[x-\frac{\nu+\alpha_1}{m+\beta_1}\right]^i\left[y-\frac{k+\alpha_2}{n+\beta_2}\right]^j\\
&\quad\times f_{x^iy^j}\left[\frac{\nu+\alpha_1}{m+\beta_1}+t\left(x-\frac{\nu+\alpha_1}{m+\beta_1}\right),\frac{k+\alpha_2}{n+\beta_2}+t\left(y-\frac{k+\alpha_2}{n+\beta_2}\right)\right]\\
&\quad\times(1-t)^{r-1}\mathrm{d}t. \qquad (4.8)
\end{aligned}$$

Taking (4.3)–(4.5) into account, (4.8) turns into the following form:

$$F(u) - \sum_{h=0}^{r}\frac{1}{h!}F^{(h)}(0)u^h = \frac{u^r}{(r-1)!}\int_0^1[F^{(r)}(tu)-F^{(r)}(0)](1-t)^{r-1}\mathrm{d}t. \quad (4.9)$$

From (4.3), (4.8), (4.9) and the fact that $F^{(r)} \in \mathrm{Lip}_M(\gamma)$ it follows that

$$\begin{aligned}
&f(x,y) - P_{r,\left(\frac{\nu+\alpha_1}{m+\beta_1},\frac{k+\alpha_2}{n+\beta_2}\right)}\left(x-\frac{\nu+\alpha_1}{m+\beta_1}, y-\frac{k+\alpha_2}{n+\beta_2}\right)\\
&= \left|F(u)-\sum_{h=0}^{r}\frac{1}{h!}F^{(h)}(0)u^h\right|\\
&\le \frac{|u|^r}{(r-1)!}\int_0^1[F^{(r)}(tu)-F^{(r)}(0)](1-t)^{r-1}\mathrm{d}t\\
&\le \frac{|u|^{r+\gamma}}{(r-1)!}MB(\gamma+1,r)\\
&\le \frac{M}{(r-1)!}\frac{\gamma}{\gamma+r}B(\gamma,r)|u|^{r+\gamma}\\
&\le \frac{M}{(r-1)!}\frac{\gamma}{\gamma+r}B(\gamma,r)\left|x-\frac{\nu+\alpha_1}{m+\beta_1}, y-\frac{k+\alpha_2}{n+\beta_2}\right|^{r+\gamma}. \qquad (4.10)
\end{aligned}$$

Hence combining (4.7) and (4.10), we obtain (4.6), which completes the proof. □

Now we take a function $g \in C(\mathscr{R}_A)$ which is given by

$$g(t,\tau) = |(x,y)-(t,\tau)|^{r+\gamma}. \quad (4.11)$$

Obviously $g(x,y) = 0$. From Theorem 3.2 it follows that

$$\|L_{m,n}(g;x,y)\|_{C(\mathscr{R}_A)} \to 0 \quad \text{as} \quad m,n\to\infty.$$

From (4.6) we arrive at the following result:

$$\|(L_{m,n}^{\alpha_i,\beta_j})^{[r]}(f;x,y) - f(x,y)\|_{C(\mathscr{R}_A)} \to 0 \quad \text{as} \quad m,n\to\infty.$$

Using (3.3) and Corollary 3.4 we get to the following results by means of Theorem 4.1.

COROLLARY 4.2.

Let $f \in C^r(\mathscr{R}_A)$ and $F^{(r)} \in \mathrm{Lip}_M(\gamma)$. Then the inequality

$$\|(L_{m,n}^{\alpha_i,\beta_j})^{[r]}(f;x,y) - f(x,y)\|_{C(\mathscr{R}_A)} \leq \frac{MB(\gamma,r)}{(r-1)!}\frac{\gamma}{\gamma+r}\frac{3}{2}w(g;\delta_{m,n})$$

holds, where $F^{(r)}, \delta_{m,n}$ and g are given by (4.5), (3.5) *and* (4.11)*, respectively.*

COROLLARY 4.3.

Let $f \in C^r(\mathscr{R}_A)$ and $F^{(r)} \in \mathrm{Lip}_M(\gamma)$, and assume that $g \in \mathrm{Lip}_{(1+A^2)^{\frac{r}{2}}}(\gamma)$ in Corollary 3.4. *Then we arrive at*

$$\|(L_{m,n}^{\alpha_i,\beta_j})^{[r]}(f;x,y) - f(x,y)\|_{C(\mathscr{R}_A)} \leq \frac{M(1+A^2)^{\frac{r}{2}}}{(r-1)!}\frac{\gamma}{\gamma+r}B(\gamma,r)\delta_{m,n}^{\gamma},$$

where $\delta_{m,n}$ is given by (3.5).

5. Weighted approximation of functions of two variables by $L_{m,n}^{\alpha_i,\beta_j}$

In this section we investigate the convergence of the sequence $\{L_{m,n}^{\alpha_i,\beta_j}\}$ mapping the weighted space C_ρ into B_{ρ_1}. We also study the rates of convergence of the sequence $\{L_{m,n}^{\alpha_i,\beta_j}\}$ defined on weighted spaces. In the rest of the article ρ will be given by $\rho(x,y) = 1+x^2+y^2$.

We first give the following important Korovkin type theorem (in weighted spaces) proved by Gadjiev in [11].

Theorem of Gadjiev. *Let $\{A_n\}$ be the sequence of linear positive operators mapping from $C_\rho(\mathbb{R}^m)$ into $B_\rho(\mathbb{R}^m), m \geqslant 1$, and satisfying the conditions*

$$\|A_n(1;\mathbf{x}) - 1\|_\rho \to 0, \quad \|A_n(t_j;\mathbf{x}) - x_j\|_\rho \to 0,\ j = 1,\ldots,m,$$
$$\|A_n(|t|^2;\mathbf{x}) - |\mathbf{x}|^2\|_\rho \to 0$$

as $n \to \infty$ for $\rho(\mathbf{x}) = 1+|\mathbf{x}|^2$, $x \in R^m$. Then there exists a function $f^* \in C_\rho(\mathbb{R}^m)$ such that $\|A_n(f^*;\mathbf{x}) - \mathbf{f}^*(\mathbf{x})\|_\rho \geqslant 1$.

By taking the result of the last theorem into account we conclude that verifying the conditions of the above theorem by the operators $L_{m,n}^{\alpha_i,\beta_j}$, $j = 1,2$, is not sufficient for $L_{m,n}^{\alpha_i,\beta_j}$ to be convergent to any function f in ρ norm. Hence we need to show the convergence in another norm for any function in $C_\rho(\mathscr{R})$. For this purpose, we now give the following lemma, which we shall use.

Lemma 5.1. *The operators $L_{m,n}^{\alpha_i,\beta_j}$ possess the following*:

(a) *$\{L_{m,n}^{\alpha_i,\beta_j}\}, m,n \in N$, is the sequence of linear positive operators from the weighted space $C_\rho(\mathscr{R})$ into the weighted space $B_\rho(\mathscr{R})$.*

(b) *The norms $\|L_{m,n}^{\alpha_i,\beta_j}\|_{C_\rho \to B_\rho}$ are uniformly bounded* (*i.e. there exists an $M > 0$ such that $\|L_{m,n}^{\alpha_i,\beta_j}\|_{C_\rho \to B_\rho} \le M$*).

Proof. From Lemma 3.1 we easily obtain that

$$|L_{m,n}^{\alpha_i,\beta_j}(\rho;x,y)| \le M(1+x^2+y^2)$$

which proves (a). Taking Lemma 3.1 into account we get the following inequality:

$$\begin{aligned}
&\|L_{m,n}^{\alpha_i,\beta_j}\|_{C_\rho \to B_\rho} \\
&\quad \le \|L_{m,n}^{\alpha_i,\beta_j}(\rho;x,y) - \rho(x,y)\|_\rho + 1 \\
&\quad \le \|L_{m,n}^{\alpha_i,\beta_j}(1;x,y) - 1\|_\rho + \|L_{m,n}^{\alpha_i,\beta_j}(t^2+\tau^2;x,y) - x^2 - y^2\|_\rho + 1 \\
&\quad = \sup_{(x,y)\in\mathscr{R}} \left| \frac{(m^2-m)x^2 + (2\alpha_1+1)mx + \alpha_1^2}{(m+\beta_1)^2} \right. \\
&\qquad \left. + \frac{n^2y^2 + (2\alpha_2+1)ny + \alpha_2^2}{(n+\beta_2)^2} - x^2 - y^2 \right| \frac{1}{1+x^2+y^2} + 1
\end{aligned} \tag{5.1}$$

so (b) is obtained from (5.1), which completes the proof. □

Now the following theorem shows the convergence of the sequence of linear positive operators $\{L_{m,n}^{\alpha_i,\beta_j}\}$, mapping from C_ρ into B_{ρ_1}, in ρ_1 norm.

Theorem 5.2. *Let $\rho_1(x,y)$ be a weight function satisfying*

$$\lim_{|x|\to\infty} \frac{\rho(x,y)}{\rho_1(x,y)} = 0. \tag{5.2}$$

Then $\|L_{m,n}^{\alpha_i,\beta_j}(f;x,y) - f(x,y)\|_{\rho_1} \to 0,\ m,n \to \infty$ for all $f \in C_\rho(\mathscr{R})$, where $x = (x,y) \in R$.

Proof. Let us denote the region $[0,1]\times[0,s], s>0$, by $\mathscr{R}_s$. Therefore we have

$$\begin{aligned}
&\|L_{m,n}^{\alpha_i,\beta_j}(f;x,y) - f(x,y)\|_{\rho_1} \\
&\quad = \sup_{(x,y)\in\mathscr{R}} \left| \frac{L_{m,n}^{\alpha_i,\beta_j}(f;x,y) - f(x,y)}{\rho_1(x,y)} \right| \\
&\quad = \sup_{(x,y)\in\mathscr{R}_s} \frac{|L_{m,n}^{\alpha_i,\beta_j}(f;x,y) - f(x,y)|}{\rho(x,y)} \frac{\rho(x,y)}{\rho_1(x,y)} \\
&\qquad + \sup_{(x,y)\in\mathscr{R}\setminus\mathscr{R}_s} \frac{|L_{m,n}^{\alpha_i,\beta_j}(f;x,y) - f(x,y)|}{\rho(x,y)} \frac{\rho(x,y)}{\rho_1(x,y)}.
\end{aligned} \tag{5.3}$$

Since ρ/ρ_1 is bounded on $\mathscr{R}_s$, the first term on the right-hand side of (5.3) approaches zero when $m,n \to \infty$ by Theorem 3.2. The second term also approaches zero when $m,n \to \infty$ by Lemma 5.1(b) and the condition (5.2). So proof is completed. $\square$

As a result we give the approximation order of $L_{m,n}^{\alpha_i,\beta_j}$, $j = 1,2, m,n \in \mathbb{N}$, by means of the weighted modulus of continuity.

Theorem 5.3. *For any $s > 0$ and all $m,n \in N$ the inequality*

$$\sup_{\|f\|_\rho=1} \left\{ \sup_{\sqrt{x^2+y^2}\le s} |L_{m,n}^{\alpha_i,\beta_j}(f;x,y) - f(x,y)| \right\} \le c \sup_{\|f\|_\rho=1} [w_\rho(f,\delta)] \tag{5.4}$$

holds for the linear positive operators $\{L_{m,n}^{\alpha_i,\beta_j}\}$, $j = 1,2$, defined on C_ρ, where $\delta = \sqrt{L_{m,n}^{\alpha_i,\beta_j}[(t-x)^2 + (\tau-y)^2]}$ and $c > 0$ is a constant depending on s.

Proof. Since $L_{m,n}^{\alpha_i,\beta_j}$, $j = 1,2, m,n \in \mathbb{N}$, are linear positive operators, we have

$$\begin{aligned}
&|L_{m,n}^{\alpha_i,\beta_j}(f;x,y) - f(x,y)| \\
&\quad \le L_{m,n}^{\alpha_i,\beta_j}(|f(t,\tau) - f(x,y)|;x,y) + |f(x,y)|(L_{m,n}^{\alpha_i,\beta_j}(1;x,y) - 1) \\
&\quad \le L_{m,n}^{\alpha_i,\beta_j}\left(\rho(x,y) w_\rho\left[f; \frac{\sqrt{(t-x)^2+(\tau-y)^2}}{\delta}\delta\right];x,y\right),
\end{aligned}$$

by Lemma 3.1. Using (2.5) we get

$$\begin{aligned}
&|L_{m,n}^{\alpha_i,\beta_j}(f;x,y) - f(x,y)| \\
&\quad \le \rho(x,y) w_\rho(f;\delta) L_{m,n}^{\alpha_i,\beta_j}\left(1 + \left[\frac{\sqrt{(t-x)^2+(\tau-y)^2}}{\delta}\right];x,y\right) \\
&\quad \le \rho(x,y) w_\rho(f;\delta) L_{m,n}^{\alpha_i,\beta_j}\left(1 + \left[\frac{(t-x)^2+(\tau-y)^2}{\delta^2}\right];x,y\right) \\
&\quad \le \rho(x,y) w_\rho(f;\delta) L_{m,n}^{\alpha_i,\beta_j}(\rho;x,y) + \frac{1}{\delta^2} L_{m,n}^{\alpha_i,\beta_j}([(t-x)^2 + (\tau-y)^2];x,y).
\end{aligned} \tag{5.5}$$

Since $(t-x)^2 + (\tau-y)^2 \in C_\rho$, (5.5) gives

$$\begin{aligned}
|L_{m,n}^{\alpha_i,\beta_j}(f;x,y) - f(x,y)| \le &\sup_{\sqrt{x^2+y^2}\le s} c^2 w_\rho(f;\delta) \|L_{m,n}^{\alpha_i,\beta_j}(\rho;x,y)\|_\rho \\
&+ \frac{1}{\delta^2} \|L_{m,n}^{\alpha_i,\beta_j}((t-x)^2 + (\tau-y)^2;x,y)\|_\rho,
\end{aligned}$$

where $c = \sup_{\sqrt{x^2+y^2}\le s} \rho(x,y)$. $\|L_{m,n}^{\alpha_i,\beta_j}(\rho;x,y)\|_\rho$ is bounded since

$$\|L_{m,n}^{\alpha_i,\beta_j}(\rho;x,y)\|_\rho = \|L_{m,n}^{\alpha_i,\beta_j}(\rho;x,y)\|_{C_\rho \to B_\rho}$$

which is uniformly bounded, for all $m,n\in\mathbb{N}$, by Lemma 5.1. From (5.5) and (5.6) we arrive at

$$\sup_{\sqrt{x^2+y^2}\leq s}|L_{m,n}^{\alpha_i,\beta_j}(f;x,y)-f(x,y)|\leq c^2(1+M)w_\rho(f;\delta),$$

which implies that

$$\sup_{\|f\|_\rho=1}\left\{\sup_{\sqrt{x^2+y^2}\leq s}|L_{m,n}^{\alpha_i,\beta_j}(f;x,y)-f(x,y)|\right\}$$
$$\leq c^2(1+M)\sup_{\|f\|_\rho=1}[w_\rho(f,\delta)],$$

where $\delta=\sqrt{\|L_{m,n}^{\alpha_i,\beta_j}([(t-x)^2+(\tau-y)^2];x,y)\|_\rho}$. Last inequality gives (5.4), which completes the proof. □

References

[1] Altın A, Doğru O and Taşdelen F, The generalization of Meyer–König and Zeller operators by generating functions, *J. Math. Anal. Appl.* (in print)

[2] Baskakov V A, On a construction of converging sequences of linear positive operators, *Studies of Modern Problems of Constructive Theory of Functions* (1961) 314–318

[3] Becker M, Global approximation theorems for Szazs–Mirakyan and Baskakov operators in polynomial weight spaces, *Indiana Univ. Math. J.* **27(1)** (1978) 127–142

[4] Bernstein S N, Démonstration du théoréme de Weierstrass fondée sur la calcules des probabilites, *Comm. Soc. Math. Charkow Sér.* **13(2)** (1912) 1–2

[5] Büyükyazıcı İ and İbikli E, The approximation properties of generalized Bernstein polynomials of two variables, *Appl. Math. Comput.* **156(2)** (2004) 367–380

[6] Callahan J, Advanced Calculus, Lecture Notes (USA: Smith College)

[7] Zhou D X, Weighted approximation by Szász–Mirakjan operators, *J. Approx. Theory* **76(3)** (1994) 393–402

[8] Doğru O, Weighted approximation of continuous functions on the all positive axis by modified linear positive operators, *Int. J. Comput. Numerical Anal. Appl.* **1(2)** (2002) 135–147

[9] Doğru O, Özarslan M A and Taşdelen F, On positive operators involving a certain class of generating functions, *Studia Scientiarum Mathematicarum Hungarica* **41(4)** (2004) 415–429

[10] Gadjiev A D, Weighted approximation of continuous functions by positive linear operators on the whole real axis, (Russian) *Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Tehn. Mat. Nauk* **5** (1975) 41–45

[11] Gadžiev A D, Positive linear operators in weighted spaces of functions of several variables, *Izv. Akad. Nauk Azerbaĭdzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk 1* **4** (1980) 32–37

[12] Hermann T, On the Szász-Mirakian operator, *Acta Math. Acad. Sci. Hungar.* **32(1–2)** (1978) 163–173

[13] Ispir N and Atakut Ç, Approximation by modified Szasz-Mirakjan operators on weighted spaces, *Proc. Indian Acad. Sci. (Math. Sci.)* **112(4)** (2002) 571–578

[14] Kirov G H, A generalization of the Bernstein polynomials, *Math. Balkanica (N.S.)* **6(2)** (1992) 147–153

[15] Kirov G H and Popova L, A generalization of the linear positive operators, *Math. Balkanica (N.S.)* **7(2)** (1993) 149–162

[16] Stancu D D, Approximation of functions by a new class of linear polynomial operators, *Rev. Roumaine Math. Pures Appl.* **13** (1968) 1173–1194

[17] Stancu D D, A new class of uniform approximating polynomial operators in two and several variables, *Proceedings of the Conference on the Constructive Theory of Functions (Approximation Theory)* (Budapest, 1969) pp. 443–455 (Budapest: Akadémiai Kiadó) (1972)

[18] Walczak Z, Approximation of functions of two variables by some linear positive operators, *Acta. Math. Univ. Comenianae* **LXXIV(1)** (2005) 37–48

[19] Volkov V I, On the convergence of sequences of linear positive operators in the space of continuous functions of two variables, (Russian) *Dokl. Akad. Nauk SSSR (N.S.)* **115** (1957) 17–19